Actions with Fractions

Fractions are a perennially difficult subject to teach to children. For many it really starts to fall apart when they need to learn operations with fractions. Frequently they use whole number thinking to make sense of fractions. Instruction on fractions is often assisted by (concrete) models. These models can be distinguished into a variety of different types: set, linear, and area models. For example, Colomb and Kennedy (2006) use set and area models to assist children in getting a better understanding of the fraction ½.

In this article we like to share our success with using array models for learning about fractions and operations with fractions. The array is an advantageous *multiplicative structure* that is applicable in many areas throughout K-12 mathematics and beyond. We will make good use of this structure in the activities to follow. The activities presented in this article address the contextual and conceptual understanding of fractions, equivalent fractions, and addition and subtraction with unit fractions. This sequence of activities, taught over four consecutive days in late spring in a fourth grade class in the Hudson Valley, demonstrates that using a context and concrete materials in an array structure may help students develop a deeper understanding of fractions, their numerical representations, and operations with (unit) fractions. We will focus on the array model using ice cube trays, egg cartons, muffin tins, and other readily available array shapes (**See figure 1**).

The context we will use here is a barn with pens that are occupied by pigs. A fraction will tell us *how full* the barn is. For example, using an ice cube tray with 12 partitions, we can say that a barn that is 11/12 filled is nearly full, one that is 2/12 filled is nearly empty, and one that is 5/12 filled is nearly half full. A fraction in its numerical

form will give us three pieces of information: How many pens there are in the barn (the denominator); how many of these pens are occupied by a pig (the numerator); and how full the barn is (the fraction proper).

Figure 1:

We will use the context of two farmers joining the contents of their barns into one barn and the contents of one barn being separated into two barns to model addition and subtraction of fractions respectively. For these operations we will limit ourselves to unit fractions. Students will be encouraged to find patterns in the addition and subtraction of unit fractions and develop their own rules. The context of the barns (arrays) is intended to assist students in realizing that the (common) denominator is not added or subtracted, a mistake often made by students. Last we aim that they will see the methodological similarity of adding and subtracting fractions.

While we used the context of barns, pens in barns, and pigs in pens, we realize that this may not be a helpful context for all children. You can replace this context with school busses, seats, and children in seats, or apartment buildings, with apartments, and families in apartments. You can use many other contexts that are relevant to your students as long as they can be represented by an array structure.

Objectives

The students will—

- Use context to give meaning to a fraction as representing how full something is
- Use context to understand that in equivalent fractions the whole is repartitioned (renamed)
- Use context to develop an understanding of addition and subtraction of unit fractions
- Discover a rule for addition and subtraction of unit fraction

Materials

A variety of arrayed trays, such as ice-cube trays and egg cartons (**See figure 1**).

Large number of centimeter cubes, at least 20 per student, or other objects that can function as "pigs."

Skewer sticks (or paper strips) cut to size to fit the width and the length of the trays.

Copies of worksheets for all students

Prior Knowledge

The students who participated in this sequence of lessons were comfortable with their multiplication facts and also had learned to see arrays as multiplicative structures through their work with the Math Trailblazers ™ curricular materials in the previous grades. They also had an understanding of factors and multiples. Students had a beginning idea of fractions and were familiar with words such as denominator and numerator. They had prior experiences with partitioning a set of objects into equally sized

groups and the identifying a subset of these groups. They had an understanding of equivalence through the experience with renaming fractions by repartitioning the whole (See: de Groot & Moone, 2006; Lamon, 2002a, 2002b, 2002c).

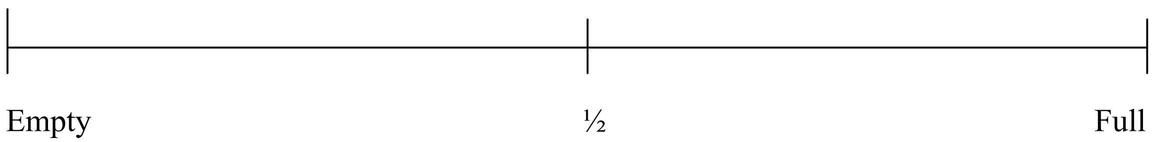
Activity One: Working with Barns

In this lesson we will establish the pigs in a barn context. Students will practice representing fractions by partitioning and re-partitioning the array structures. They will also learn that the denominator represents the number of equally sized groups of pens, the numerator identifies how many of those groups are occupied by pigs, and the fraction symbol as a whole represents the fullness of the barn. We used a variety of array structures that we purchased in several department stores. We had 2x6, 2x8, and 6x10 ice cube trays; 2x6 egg crates; 3x4 and 4x6 muffin tins; and a 4x5 cup tray we found in the school cafeteria (**See figure 1**). These arrays represent barns with total number of pens of 12, 16, 20, 24, and 60 pens. Some of these barn sizes have many factors. We were keen on this to assist the children in discovering that the factors of the barn size help them determine in which ways they can re-partition the array. For example, a barn with 12 pens can be re-partitioned as a barn with two groups of six pens (2x6), creating halves, or six groups of two pens (6x2), creating sixths. This is an important scaffold to the idea of a common denominator.

Provide each student with a "barn" such that students near each other have different sized barns. Also give each student as many centimeter cubes (or other object representing pigs) as partitions in the array. Ask students to determine how many pens are in their barn and to make sure they have enough cubes. Introduce the rules for putting

pigs in the pens: 1. Only one pig per pen, and 2. Place pigs in pens as close together as possible (i.e. adjacent pens). These rules are important for the renaming process.

Figure 2:



Assign the students to create fractions with their barns and record these fractions on a number line (**See figure 2**) that spans the width of a regular sheet of paper between 0 and 1 (empty and full). We found that most students eventually recorded all possible fractions with the denominator equal to the number of partitions in their array.

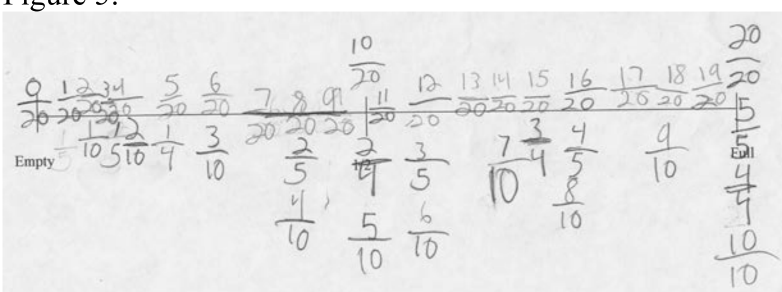
Figure 3:

Figure 4

Since these students had learned about renaming in a previous experience (Authors, 2006), we challenged them to rename the fractions they recorded when possible. You may need to model this with your students before you pose such a challenge. We provided the students with skewer sticks that assisted them in repartitioning the arrays. For example, in **figure 3** you can see how a student renamed 4/12 as 2/6 and in **figure 4**

you can see how two students renamed 20/24 as 5/6. You can also start encouraging students to draw their fractions, and construct the repartitioning, on paper using the worksheet in **Appendix A**. The next step is to challenge students to rename until no further renaming can be done, such as renaming the fraction in **figure 3** to 1/3. This was all recorded on the number line (**See figure 5**). This is an important part of the lesson. Students will find that certain fractions cannot be renamed. For example in the 12-pen barn, 5/12, 7/12, and 11/12 cannot be renamed. As a last challenge, ask students to explain why some fractions can and others can't be renamed.

Figure 5:



Finish the lesson with a sharing period where you focus on why certain fractions cannot be simplified (a more common term for renaming). The objective is to come to the point where students realize that the number of pens occupied by a pig (the numerator) and the number of pens in the barn (the denominator) must have a factor in common to rename the fraction. This fundamental idea needs to be reinforced throughout the remainder of this sequence of lessons. One student offered that 11/12 couldn't be renamed because the 11 is not even. We replied to her that she did rename 3/12 but that the 3 is not even either. After a brief moment she replied, "Yes, but three is a factor of 12 and 11 is not!" Then we asked the class if others found the same idea. And several students gave examples with other denominators. Then we returned to the first student

and asked her to explain why she could rename 9/12 to 3/4, because, we posed, 9 is not a factor of 12 either. She explained that 9 and 12 both have a factor of 3! She then turned around to a student who had a 60-pen tray and stated, "I can tell you which fractions can be renamed for his tray; they need to have a factor of 60, like 3, 15, 20, 6, 10, 4, and some more." Other students discovered patterns such as, "In my 20-pen barn tenths show up every two twentieths and fifths show up every four twentieths, and all fifths are also tenths."

We are now ready to move to the next lesson, where we will deepen the importance of the multiplicative structure. We will do this at the representational level by using array pictures, such as in **Appendix A**.

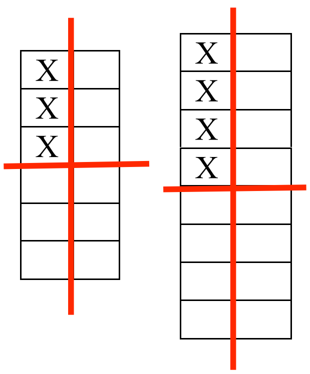
Activity Two: Deepening the Multiplicative Structure

Prepare the lesson by making an overhead and handouts of **Appendix B**. Before you begin this lesson you may want to review the meanings of factors and multiples. We noticed that children would use these words sometimes interchangeably, while they did seem to give correct explanations. We dealt with this by modeling the correct language rather than turning it into a vocabulary lesson. This was very effective and didn't disturb their flow of thought. This lesson prepares the idea of a common denominator (same sized barns) for addition and subtraction.

Have students work in pairs and give each pair two different arrays with sufficient cubes and skewer sticks. Also provide them with another handout of **Appendix A**. Tell them that they should determine which (renamed) fractions they can both make with their different barns. Assign them to demonstrate this using a picture representation using

Appendix A. During the work, challenge students whether, for example, ¼ of a 12-pen barn (3/12) and ¼ of a 16-pen barn (4/16) represent the same quantity (**See figure 7**). Students can reason that while both represent ¼ of a barn, the two barns are not the same size. If two farmers would exchange ¼ of these barns it would not be fair. This emphasizes that exchanging or adding and subtracting fractional parts can only be done if it is "fair."

Figure 7:



After students have had sufficient time to record their findings, have a class discussion. Begin with some interesting examples you noticed during the group work. Then move the discussion to **Appendix B**. Inventory systematically with the students which size barns can have fractions renamed to those in the left column. The idea is that students realize that in the right column only barns occur with sizes that are a multiple of the denominator (the number of partitions). Ask students why sevenths was not put in the table. Students in this project said, "The barn would have to be 14, or 21, or 28, and seven is not a factor of any of the barn numbers. Discuss and stress, using the pictorial representations, that, for example, the thirds from all the different barn sizes cannot be exchanged fairly. Farmers can only exchange the same parts from the same sized barns fairly.

Activity Three: Problem Solving with Fractions

In this lesson the children will solve an addition problem and a subtraction problem. They were asked to show their work in pictures, numbers, and words (**See Appendix C**). We will describe the process for the addition problem here. The process for the subtraction problem is identical, but should be done after addition is completed in lesson 4.

Give each student a 12-pen barn and 12 pigs. Provide them with **Appendix C** first.

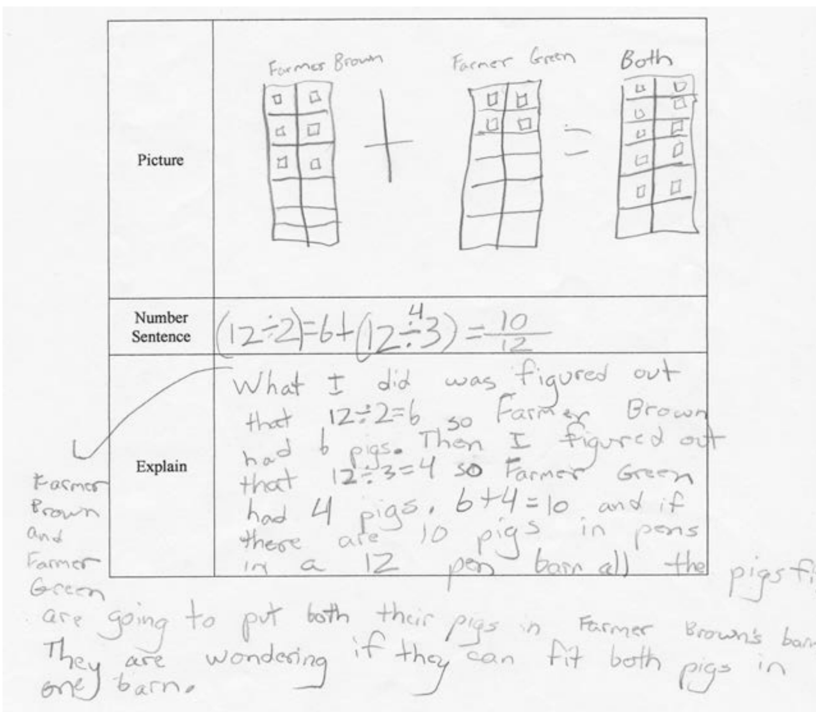
To assist children with reading this complex problem we posed the following questions in the order presented to guide the discussion:

- What does the question ask? Does the question ask for how many pigs are in the barn altogether?
- Does the problem tell you how many pigs should go in each barn?
- Why did we give you a barn that has 12 pens in it for this problem? What does the number of pens in the barn have to do with ½ and 1/3?
- Could we have used barns with a different number of pens? If so which ones?
- Why do both farmers' barns need to have the same number of pens?

In this discussion it is important that you keep focusing the students on the barn. The barn is the whole here, the number of pens or equally sized groups of pens represents the denominator, and the number of pens or groups of pens that have pigs in them represents the numerator.

The participating students realized that they were not told how many pigs were in each barn, but that they knew how full each barn was. This linked back to the concept of fractions representing fullness. However, they quickly stated that a half barn had six pens filled and a third barn had four pens filled. Students started to realize that they now could join the pigs from one barn with those in the other barn to do the addition. We had the students act this out in pairs and complete the worksheet (**See Appendix C and figure 10**).

Figure 10:



During the sharing part of the lesson return to the previously listed questions. Students in this project spontaneously said that other arrays could have been used, such as a 24-pen, a 60-pen, and other sizes that we did not have samples of (e.g. 18-pen barn). One of the students suggested using one 12-pen barn and one 18-pen barn. When this was modeled on the board using array pictures one student explained that, "You can make the

same fraction in both barns, but it is not the same number of pigs. Half of farmer Green's barn is six pigs and half of farmer Brown's barn is nine. If farmer Green and farmer Brown would trade a half a barn it would not be fair." The students agreed that you must use the same size barns for these problems. It is very important for students to make pictorial representations at this stage. Otherwise they may fall back on using (often incorrectly) whole number sense.

One student made a remarkable discovery. She said, "You can't use 16 because it can't be made into thirds. Even a six-pen barn would have worked! You basically need to have *an even multiple of three*, and 16 is not an even multiple of 3." This led us to ask the class what the smallest sized barn was that could help us solve this problem. The class quickly agreed this had to be six. We used egg cartons to demonstrate this (**See figure 11**).

Figure 11:

Before After

4/12	6/12	10/12	2/6	3/6	5/6
½	1/3	5/6	1/2	1/3	5/6

Egg cartons are handy because you can easily cut these into smaller "barns" using a serrated knife (make sure you put the open side on the table or board when you cut). This beginning idea of a lowest common denominator returned in the next lesson where we asked students to repeat the problem for different parts of barns.

Activity Four: Addition and Subtraction, Patterns and Rules

In preparation for this lesson make an overhead and sufficient copies of **Appendix A and Appendix D**. Students will act out addition (and subtraction) of unit fractions using the array structures. During this lesson we were promoting students to use the array pictures, but we kept the trays and cartons available. A few students did have a preference to continue using the concrete materials. The students used these materials to derive sums (and differences), for a variety of problems that use unit fractions only and are presented more or less in a pattern so that students may discover a rule for adding and subtracting fractions (**See figures 13, 14, and 16**). We avoided addends where denominators had factors in common at this time. We also left two blank options in the worksheets. This proved to be a good assessment tool as students started to use it to check their conjectures. Several students actually ran into improper fractions on without our assistance (**See figure 14**).

Figure 13:

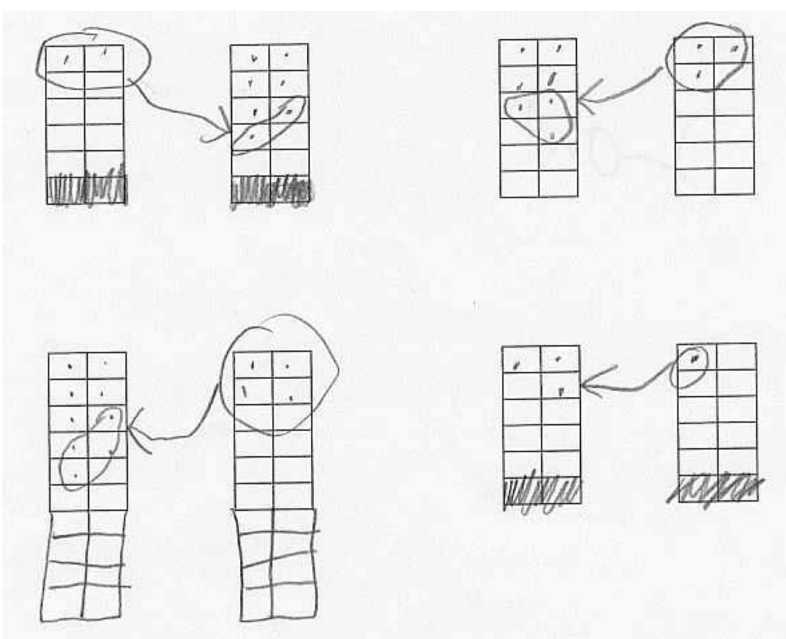
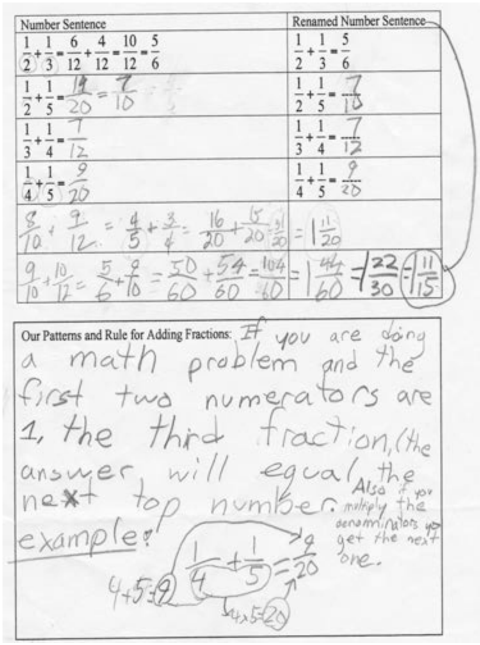
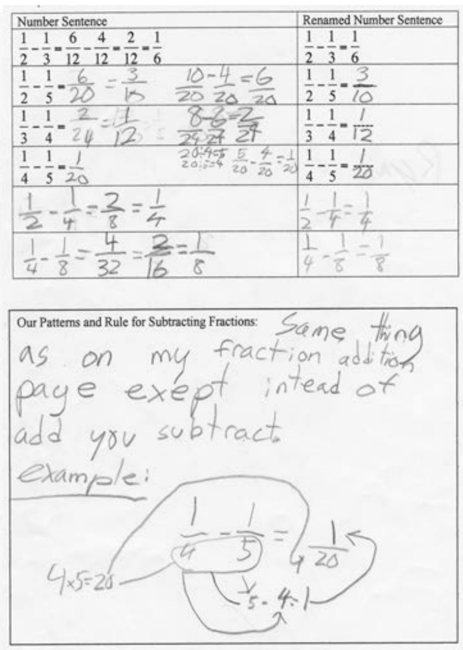
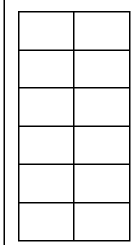
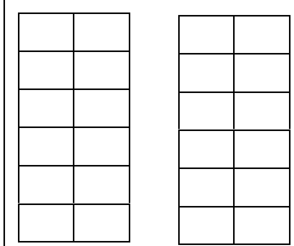
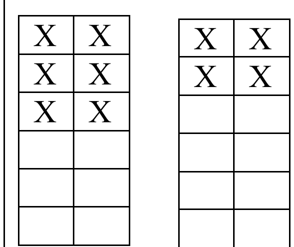
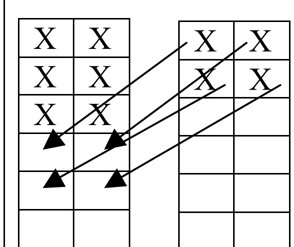
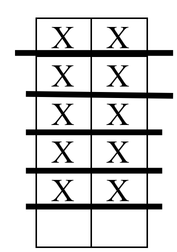


Figure 14: Figure 16:





Begin the lesson with discussing the worked example of the worksheet. It relates directly to the farmers problem. Identify to the children that the right column should be used to discover a rule for adding (subtracting) these fractions. In order to assist the children with these problems we promoted a five-step process as follows:

Step	Rule	Student comments	Pictures
1. Identify the number of pens needed in each barn.	1. The number of pens needed is a multiple of both denominators.	Take the 12-pen barn, because 2 and 3 are both factors of 12.	
2. Make the barns with the pens.	2. Each barn must have the same number of pens.	Make two barns: One for farmer Green and one for farmer Brown, each with 12 pens, which we got from step one.	
3. Fill in the parts of each farmer's barn.	3. The fractions tell you how many equal groups you have to make (denominator) and fill with pigs (numerator).	If we take 12-pen barns then the one that is half full has six pens filled with pigs and the one that is one third full has four pens filled with pigs.	
4. Move the pigs from one barn to the other barn. Use arrows in your pictures to show how you did that.	4. If all the pens of the first barn are filled with pigs, the rest stay in the second barn, because there can only be one pig in a pen.	You have to combine them, because that's what it says in the problem. You move four pigs from one barn with the six from the other barn.	
5. Draw the final result, rename, and record how full the first barn is now.	5. Show just the combined result. Make sure the result is renamed until you cannot make it simpler.	In the combined result we renamed the result to sixths. You can take half of 10 and 12. We could have made it easier by starting with a 6-pen barn.	

After this introduction challenge students to complete **Appendix D**. We suggest having students work in pairs. While you observe the students at work keep them focused on the procedure described above and also challenge them to find a rule.

Make sure to have sufficient time to have a class discussion on finding the rule. Some of the students noticed, when working with fractions with numerators of 1, that the resulting sum of the fractions had a pattern to it. If they added the value of the 2 denominators, it equaled the new numerator. And likewise if they multiplied the value of the denominators, it would equal the new denominator. Some answers still needed to be renamed, and the students did so. As this rule was revealed, the students eagerly tried it out to see if it really works. It was interesting to note that the students realized this rule only worked if the numerator was one, and began to look for a pattern or rule when the numerators in the problem were greater than one. Fractions were no longer fearful!

As a last section in this sequence we invited students to try to find a rule for subtraction. We provided the following context for subtraction: *Farmer Brown has ½ of a barn filled with pigs and decides to sell 1/3 of that barn to farmer Green. What part of farmer's Brown barn is filled after the sale?* We provided the students with **Appendix E**. Notice how we used the same fractions and changed the operation from addition to subtraction. In the class discussion students realized that subtractions of fractions goes almost the same as the addition, except "you are taking away and not combining." (**See figure 16**).

Beyond the Lesson

In this sequence of lessons the students discovered a rule for adding and subtracting *unit* fractions: $\frac{1}{a} \pm \frac{1}{b} = \frac{b \pm a}{a \times b}$. In words: The numerator of the answer is the

sum (difference) of the addends' denominators and the denominator is the product of the addends' denominators. Note that the numerator is $b \pm a$. This is not obvious when adding due to the commutative property, but in subtraction we subtract *a* from *b*.

You can extend the sequence to include fractions other then unit fractions. First leave one of the addends a unit fraction and then make both addends fractions other than unit fractions. This may need some time, but students may discover that the sum of two unit fractions is a special case of the more general rule:

$\frac{1}{a} \pm \frac{1}{b} = \frac{(1 \times b) \pm (1 \times a)}{a \times b}$ comes from $\frac{c}{a} \pm \frac{d}{b} = \frac{(c \times b) \pm (d \times a)}{a \times b}$, where c = d = 1.

In fact several students did find this out during this project without prompting.

In this project we limited the activities to sums less than or equal to one. That is, all the pigs always ended up in one barn. You can, of course, use this same model to explore problems in which more than one barn will be filled after the pigs are joined. This will initially lead to "improper" fractions. An improper fraction can be understood in this context as signifying that more than one barn is filled. Again, several students discovered improper fractions on their own without teacher prompts.

Last, the array model is also very well suited to help students conceptually understand multiplication and division of fractions (see, for example, Flores and Klein, 2005; Van de Walle, 2004).

Concluding remarks/reflections

When we started this project students weren't particularly thrilled that it was going to be about fractions. Some of them said they felt nervous when they heard this. At the end of the sequence we checked what had happened to these feelings. All students

raised their hands when asked if they felt more positively about fractions. They particularly identified that the contextual aspects of the work made it more fun and less threatening. They were very excited about the discovered rules and how it made everything so simple.

Our general approach in this sequence of lessons has been to provide a concrete context (pigs in pens in barns) that the students can relate to and that have the potential to contextualize (make real) all mathematical concepts involved in operations with fractions. We were particularly keen on revealing the underlying multiplicative relationships. To this end we helped the students go from barns to arrays, for which we used a pictorial model. Then from this modeling we moved the students to regarding fractions from a more abstract numerical point of view by asking them to develop a rule. For the majority of these forth graders this was, in our judgment, extremely successful.

References

Colomb, Joanne, and Kim Kennedy. Your Better Half. *Teaching Children Mathematics* 12 (4) (2005): 180-190.

Flores, Alfinio, and Erika Klein. "From Students' Problem-Solving Strategies to Connections in Fractions." *Teaching Children Mathematics* 11 (9) (2005): 452-57.

Lamon, Susan J., Part-whole comparisons with unitizing, in *Making sense of fractions, ratios, and proportions,* Litwiller, Bonnie, and George Bright (Eds.), Reston, VA: NCTM (2002a): 79-86.

Lamon, Susan J., Rename it, in *Classroom activities for Making sense of fractions, ratios, and proportions,* Bright, George, and Bonnie Litwiller (Eds.), Reston, VA: NCTM (2002b): 14-16.

Lamon, Susan J., Can you see it?, in *Classroom activities for Making sense of fractions, ratios, and proportions,* Bright, George, and Bonnie Litwiller (Eds.), Reston, VA: NCTM (2002c): 17-19.

Moone, G., and C. de Groot (2006). Fraction action. *Teaching Children Mathematics.* Reston, VA: National Council of Teachers of Mathematics, *(13)*, 5, 266-271.

Van de Walle, John. Elementary and Middle School Mathematics: Teaching Developmentally. New York: Pearson, 2004.

Appendix A

Appendix B

Renamed fractions	Number of pens in your barn
Thirds $\frac{\dots}{3}$	
Fourths $\frac{\dots}{4}$	
Fifths $\frac{\dots}{5}$	
Sixths $\frac{\dots}{6}$	
Eights $\frac{\dots}{8}$	
Tenths $\frac{\dots}{10}$	

Renamed fractions	Number of pens in your barn
Thirds $\frac{\dots}{3}$	12, 24, 60
Fourths $\frac{\dots}{4}$	12, 16, 20, 24, 60
Fifths $\frac{\dots}{5}$	20, 60
Sixths $\frac{\dots}{6}$	12, 24, 60
Eights $\frac{\dots}{8}$	16, 24
Tenths $\frac{\dots}{10}$	20, 60

Appendix C

Barn Fractions

Your Name: ____

With a partner use your ice cube trays to solve the following problem. Then make a drawing of how you used your trays. Next write a number sentence for this problem. Last, write a paragraph explaining how you solved this problem.

Farmer Brown and Farmer Green, who are neighbors, agree that during the winter they will put their pigs in one barn to keep them warm. Both farmers have the same sized barns with the same number of pens in each. Farmer Brown has ½ of a barn filled with pigs and Farmer Green has 1/3 of a barn filled with pigs. After all the pigs from Farmer Green's barn are put in Farmer's Brown's barn, then what part of the first barn is filled?

Picture	
Number Sentence	
Explain	

Appendix D

Use your ice cube trays and other arrays to solve the pig farmer's problem.
For each problem rename the answer until you cannot rename it any further. While you are doing this work with a partner, discuss any patterns that you notice with the numbers in each number sentence. Write down what patterns you noticed and then write down a rule for adding these kinds of fractions.

Number Sentence	Renamed Number Sentence
$\frac{1}{2}+\frac{1}{3}=\frac{6}{12}+\frac{4}{12}=\frac{10}{12}=\frac{5}{6}$	$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$
$\frac{1}{2}+\frac{1}{5}=$	$\frac{1}{2}+\frac{1}{5}=$ -----
$\frac{1}{3}+\frac{1}{4}=$	$\frac{1}{3}+\frac{1}{4}=$ -----
$\frac{1}{4}+\frac{1}{5}=$	$\frac{1}{4}+\frac{1}{5}=$ -----

Our Patterns and Rule for Adding Fractions:

Appendix E

Use your ice cube trays and other arrays to solve the pig farmer's problem.
For each problem rename the answer until you cannot rename it any further. While you are doing this work with a partner, discuss any patterns that you notice with the numbers in each number sentence. Write down what patterns you noticed and then write down a rule for subtracting these kinds of fractions.

Number Sentence	Renamed Number Sentence
$\frac{1}{2}-\frac{1}{3}=\frac{6}{12}-\frac{4}{12}=\frac{2}{12}=\frac{1}{6}$	$\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$
$\frac{1}{2}-\frac{1}{5}=$	$\frac{1}{2}-\frac{1}{5}=$ -----
$\frac{1}{3}-\frac{1}{4}=$	$\frac{1}{3}-\frac{1}{4}=$ -----
$\frac{1}{4}-\frac{1}{5}=$	$\frac{1}{4}-\frac{1}{5}=$ -----

Our Patterns and Rule for Subtracting Fractions: